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Random-walk renormalisation and the spectral dimension of harmonic models on a class of hierarchical lattice

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Abstract. Random-walk recursion relations are developed on a simple example of hierarchical lattice and are found to be the generating functions for first-passage walks. With exact renormalisation of a general harmonic model the spectral dimension, F, for the bond hierarchy class is shown to be

$$F = 2\ln(g)/[\ln(g)+\ln(\lambda_r)],$$

where g and λ_r are respectively the aggregation number and resistance eigenvalue. The behaviour of F across families of hierarchies is discussed. It is noted that the renormalisation group has limitations in its abilities to give results for the above models on hierarchies and that F so found may not maintain on hierarchies its usual significance for random-walk statistics.

1. Introduction

On fractal lattices several authors (Dhar 1977, Rammal and Toulouse 1983, Alexander and Orbach 1982) have introduced the spectral dimension denoted F in this work. This parameter enters harmonic and diffusive physics: the low-frequency density of states of harmonic models obeys the homogeneity $\rho(\omega) = \lambda^{1-F}\rho(\lambda\omega)$, generalising the Euclidean power law $\rho(\omega) \propto \omega^{d-1}$.

Hierarchical lattices are a vast class of lattices possessing tractable renormalisation transformations. Bond hierarchies are studied in this work. Some discussion of the structure of bond hierarchies and corresponding terminology is given in § 2. In § 3 renormalisation of random-walk recursion relations on the Wheatstone-bridge hierarchy is presented by way of a worked example; a fixed point eigenvalue is associated with first-passage times. Section 4 shows the equivalence of the random-walk renormalisation to that of harmonic vibrations with masses scaled with site coordinations. Straightforward matrix algebra on a general form leads to the result stated in the abstract. Section 5 briefly discusses limitations of renormalisation results on hierarchies and gives a critique of the relevance of F as found in § 4 to random-walk statistics.

2. Comments on hierarchical lattice structure

Griffiths and Kaufman (1982) give a specific definition of hierarchical lattices. Many varieties of such lattices exist; the class of bond hierarchies, as previously described

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by Melrose (1983b, 1985), are studied here. The terminology and definitions of lattice metric, intrinsic dimension D and connectivity contained in the authors earlier works are assumed here; $D = \ln(g)/\ln(b)$, where g is the number of bonds on the basic cell and b is the number of bonds on the shortest path between the nodes of the basic cell (the $n = 1$ unit).

A parameter introduced here is w, the coordination of the nodes on the basic cell. In general, hierarchies can possess an infinite range of site coordinations. An nth unit has two nodes of coordination w^n and other sites with coordinations in the range k_i to $k_i w^{n-1}$, where the k_i are independent of n and are the set of coordinations on the basic cell. Sites of coordination $k_i w^m$ will be referred to as *mth-order* sites. As one constructs hierarchies one observes the following features. If $b > 1$ sites with higher and higher coordination are further and further apart. An mth-order site has, for $j < m$, jth-order sites at a distance b^j from itself. The structure surrounding an mth-order site is the same for all mth-order sites out to a distance b^m: it is surrounded by k_{i_m} mth units. At and beyond a distance b^m the structure surrounding any particular mth-order site depends on the location of this site with respect to $(p > m)$-order sites; it is clear that coordination does not exhaust division of sites into their infinite variety of equivalence classes (Griffiths and Kaufman 1982). The relevance of these structural features to random-walk statistics is discussed in § 6.

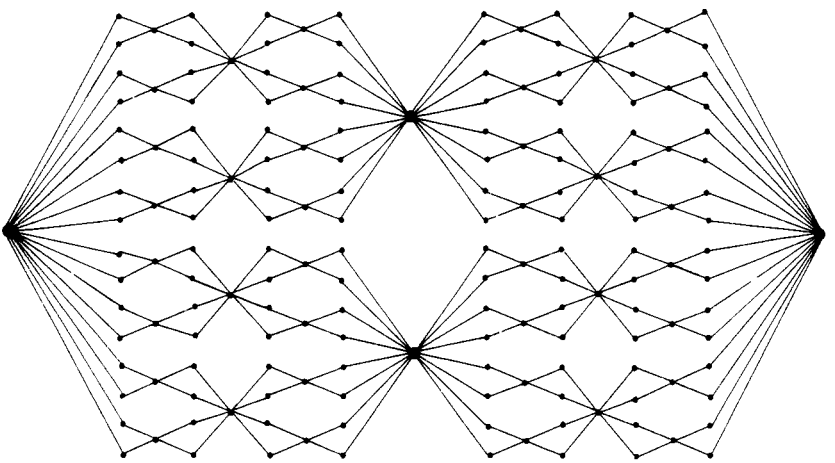

Figure 1. The fourth unit in the generation of the $A = 2$, $M = 2$ Migdal–Kadanoff hierarchy; the basic cell is that of (1a) in figure 3 in the case stated.

3. Renormalisation of random walks

The generating function for random-walk site probabilities (Montroll and Weiss 1965) is written

$$G_\alpha(\beta, Z) = \sum_n P_\alpha(\beta, N) Z^N, \tag{1}$$

where $P_\alpha(\beta, N)$ is the probability of arrival at a site β at step N having started the walk at a site α and Z^N is an ensemble weight for walks of N steps. Renormalisation is carried out using the recurrence relation obeyed by G's at neighbouring sites; an example will be worked through below.

Example. The Wheatstone-bridge hierarchy

Consider as origin of the walk some site α with coordination c_α; this is surrounded by $c_\alpha/2$ basic cells as shown in figure 2. The following relation is found to hold between the $P_\alpha(\beta, N)$:

$$P_\alpha(\alpha, N) = \tfrac{1}{3} \sum_{i=1}^{c_\alpha/2} (P_\alpha(\alpha i_1, N-1) + P_\alpha(\alpha i_2, N-1)), \tag{2}$$

where αi_1 and αi_2 denote the two neighbours of α in the ith of the $c_\alpha/2$ basic cells surrounding α (see figure 2); the probability of stepping along any bond from a given site of coordination c is $1/c$. Using (1) one finds, with $P_\alpha(\beta, 0) = \delta_{\alpha\beta}$, the recurrence relation

$$G_\alpha(\alpha, Z) - \frac{Z}{3} \sum_{i=1}^{c_\alpha/2} (G_\alpha(\alpha i_1, Z) + G_\alpha(\alpha i_2, Z)) = 1 \tag{3}$$

and similarly

$$G_\alpha(\alpha i_1, Z) - \frac{Z}{c} G_\alpha(\alpha, Z) - \frac{Z}{3} G_\alpha(\alpha i_2, Z) - \frac{Z}{6} G_\alpha(\beta i, Z) = 0 \tag{4}$$

where βi denotes the second neighbours of α in cell i; these are found to have coordination 6. Renormalisation, eliminating the zeroth-order sites, carried through by solving (4) for the $G_\alpha(\alpha i_1, Z)$ and substitution into (3) gives

$$[1 - Z^2/(3-Z)] G_\alpha(\alpha, Z) - \tfrac{1}{3}[Z^2/(3-Z)] \sum_{i=1}^{c_\alpha/2} G_\alpha(\beta i, Z) = 1. \tag{5}$$

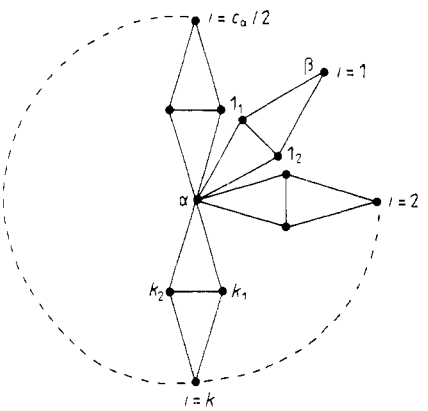

Figure 2. General origin on a WBH.

The factor $[1 - Z^2/(3-Z)]$ may be divided through to bring back the original model. The renormalisation is completed by relabelling the sites: all sites have dropped in coordination by $\frac{1}{2}$ and equation (5) is rearranged to give

$$G_{\alpha'}(\alpha', Z') - \frac{Z'}{3} \sum_{i=1}^{c_\alpha/4} (G_{\alpha'}(\alpha' i_1, Z') + G_{\alpha'}(\alpha' i_2, Z')) = [1 - Z^2/(3-Z)]^{-1}, \tag{6}$$

where

$$Z' = Z^2/(3 - Z - Z^2) \tag{7}$$

and α' denotes that α is now of coordination $c_\alpha/2$. Z' is the renormalised walk weight and the $G\beta i$ have been relabelled as the nearest neighbours of α'. Note G_α and $G_{\alpha'}$ are different function being defined for inequivalent sites.

Hoye and Napiorkowski (1980) and Napiorkowski (1983) have considered renormalisation of random walks in one dimension by making block transformations between walk graphs. Expansion of (7) and graphical interpretation of successive terms reveals that (7) is the generating function for first-passage walks from any origin to any one of the nodes of the surrounding basic cells, the βi of figure 2. Similarly the factor on the right of (6) is the generating function for walks which return to the origin without passing through any of the βi. For a pth-order origin, iteration of the recursion relation and the product of successive factors on the right of (6) give for the mth iteration the generating functions respectively for first-passage walks out to any one of the nodes of the surrounding m units and for walks that return to the origin without passing through any m unit nodes, where $m \leq p$; a discussion of the bound is given in § 6.

For a general bond hierarchy let Z'_m stand for the mth iterate of the recursion relation

$$Z'_m = \sum_L Q_{mL} Z^L. \tag{8}$$

Following the example above it is assumed that in general Q_{mL} is the probability of first passage out to any of the nodes of the surrounding mth units, where the origin is of pth order with $p > m$. The recursion relations have an unstable fixed point at $Z'_m = 1 \forall m$ (see § 4). Expectations of the generating functions Z'_m are taken at this fixed point (Feller 1950). The expectation of the number of steps required to make first passage to mth unit nodes is given by

$$\langle n \text{ step} \rangle_m = \lambda_Z^m, \tag{9}$$

where $\lambda_Z = \mathrm{d}Z'_1/\mathrm{d}Z \,|\, Z = 1$ is the fixed point eigenvalue. The walk exponent ν is defined by

$$\langle R(N) \rangle = N^\nu, \tag{10}$$

where $\langle R(N) \rangle$ is the average distance from the origin at step N. Assuming that on average first passage out to a distance R occurs at step $R^{1/\nu}$ (Angles d'Auriac *et al* 1983) one has

$$(b^m)^{1/\nu} = \lambda_Z^m \tag{11}$$

and hence

$$\nu = \ln(b)/\ln(\lambda_Z). \tag{12}$$

Similar arguments have been given by Given and Mandelbrot (1983).

4. Harmonic models and a general result for F

The spectral dimension can be calculated by the renormalisation of the Gaussian model partition function (Dhar 1977, Melrose 1983a) or that of harmonic vibrations (Rammal and Toulouse 1983, Rammal 1984). To maintain model invariance under renormalisation it is necessary that site terms in these models (masses or external fields) be applied

to sites in proportion to their coordinations (Yeomans and Fisher 1981, Melrose 1983b and references therein). Harmonic equations of motion for frequency ω are written

$$c_i(\omega/\omega_0)^2 U_i - \sum_{j=1}^{c_i} (U_j - U_i) = 0 \tag{13}$$

where $\omega_0 = (k/m)^{1/2}$ with k the spring constant and a site i of coordination c_i is given a mass $c_i m$. (On discrete spin models this site term scaling leads to an infinite susceptibility to external fields at high temperatures (Kaufman and Griffiths 1983, McKay and Berker 1984); even if one starts with constant site terms the renormalisation introduces site terms proportional to the coordination.)

As was seen in § 3 random-walk generating functions on hierarchies obey recurrence relations of the form

$$G_i - \sum_{j=1}^{c_i} \frac{Z}{c_j} G_j = \delta_{i0}. \tag{14}$$

Substituting $U_i = G_i/c_i$ one sees that the LHS's of (13) and (14) are equivalent if one sets

$$Z^{-1} = (\omega/\omega_0)^2 + 1. \tag{15}$$

The δ function on the RHS of (14) plays no role in the derivation of recursion relations and the two models have recursion relations related by the substitution (15). (Note the site term scaling of the harmonic models is now seen to be necessary to achieve the above equivalence with the random-walk model in whose equations the coordination appears quite naturally.) The notation $a = Z^{-1} = (\omega/\omega_0)^2 + 1$ is introduced and used below.

For the purposes of unity consider renormalisation of the general equations

$$MU = 0, \tag{16}$$

where U is a vector of site amplitudes U_i and M is the adjacency matrix defined by

$$M_{ii} = c_i a \qquad M_{ij} = -\sigma_{ij}, \tag{17}$$

where $\sigma_{ij} = 1(0)$ if sites i and j are (are not) neighbours. Consider a general symmetric bond hierarchy (Melrose 1985). For a particular site i the equations given by row i of (16) can be divided into c_i/w separate equations for each neighbouring cell. Renormalisation can be carried out independently on each such set of cell equations, that is (16) on a basic cell. Let n stand for the number of sites on the basic cell. The nodes are assigned as the first and second elements of U and M. Renormalisation is carried through by solving (16) for the internal sites $U_3 \ldots U_n$ in terms of the nodes U_1 and U_2 and then substitution of these solutions into the equations for U_1 and U_2, the first two rows of M. The cell is reduced to a single bond between the nodes. Let B be the resulting (2×2) matrix satisfying:

$$\begin{pmatrix} f_1(a) & -f_2(a) \\ -f_2(a) & f_1(a) \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = 0. \tag{18}$$

One finds

$$B = (M_{3\ldots n,3\ldots n}) - (M_{12,3\ldots n})(M_{12,12})^{-1}(M_{12,3\ldots n})^{\mathrm{t}} \tag{19}$$

where $(M_{i_1\ldots i_p,j_1\ldots j_q})$ denotes the matrix formed from M by elimination of rows $i_1 \ldots i_p$ and columns $j_1 \ldots j_q$ and $(A)^{\mathrm{t}}$ denotes the transpose of A. Now (18) are the equations for a single bond on the renormalised lattice. To complete the renormalisation one

rescales the U's such that the off-diagonal elements of (18) are -1 as in (17) leading to the recursion relation for a:

$$a' = f_1(a)/f_2(a), \tag{20}$$

with $a = 1$ always a fixed point (see the appendix).

As $\omega \to 0$ the density of states of harmonic vibrations (per ω per bond) obeys $\rho(\omega) = \lambda^{1-F}\rho(\lambda\omega)$ (Rammal and Toulouse 1983) and one finds

$$F = 2\ln(g)/\ln(\lambda_a), \tag{21}$$

where $\lambda_a = \mathrm{d}a'/\mathrm{d}a(a=1)$, $\lambda_a = \lambda_Z$ by (15). Now when $a = 1$ equations (16) are just Kirchoffs' laws with the bonds as unit resistors and renormalisation gives the effective resistance, λ_r, between the nodes of a basic cell, $\lambda_r^{-1} = f_1(1) = f_2(1)$; Stinchcombe (1979) noted this equivalence in the context of the Gaussian model. It is shown in the appendix that

$$\lambda_a = g\lambda_r \tag{22}$$

where g is the aggregation number (equation (19) may be written in terms of matrix determinants which on expansion of δ, where $a = 1 + \delta$, and use of (17) on the resulting expressions leads to the result (22)). From (22) one has the result stated in the abstract

$$F = 2\ln(g)/[\ln(g) + \ln(\lambda_r)]. \tag{23}$$

Given and Mandelbrot (1983) find that (22) holds on a number of hierarchical fractal lattices. (Trivially the hierarchies here do obtain scaling behaviour immediately so difficulties discussed in Given and Mandelbrot (1984) do not arise). Hilfer and Blumen (1984) have used similar matrix algebra to that above on the Sierpinski gasket.

5. Variation of F on examples

Table 1 gives values of F for the cells of figure 3 as studied previously by Melrose (1983b, 1985); notation follows these works. Hierarchical lattices are by definition finitely ramified with respect to sites. Hierarchical lattices which are also fractal with equality of lattice and Euclidean metric are also finitely ramified with respect to bonds and hence have $\lambda_r > 1$ and $F < 2$ by (23). However, the hierarchies of table 1 are infinitely ramified with respect to bonds and can have $F > 2$. On a d-dimensional regular lattice the resistance, λ_r, between two $d-1$ faces a distance L apart obeys $\lambda_r \propto L^{2-d}$. On some cells (e.g. 1a, 2a, 2b, 3a, 3b, 3d-4b) termed *uniform* one finds the

Table 1. Lattice parameters and F.

	2c	2b	2e	3c	2a	3d	3e	3a	3b	2d	4a	4b
g	12	8	16	12	5	15	20	9	12	12	14	28
q	2	2	2	3	2	3	4	3	4	3	4	8
b	4	2	4	3	2	3	3	2	2	2	2	2
λ_r	3/2	3/2	3/2	5/6	1	1	3/4	3/2	1/2	8/10	1/2	1/4
Q	0.5	0.63	0.5	1	1	1	1.26	1.58	2	1.58	2	3
F	1.72	1.67	1.74	2.16	2	2	2.21	1.69	2.77	2.20	2.71	3.42
D	1.79	1.89	2	2.26	2.32	2.46	2.73	3.17	3.58	3.58	3.81	4.81

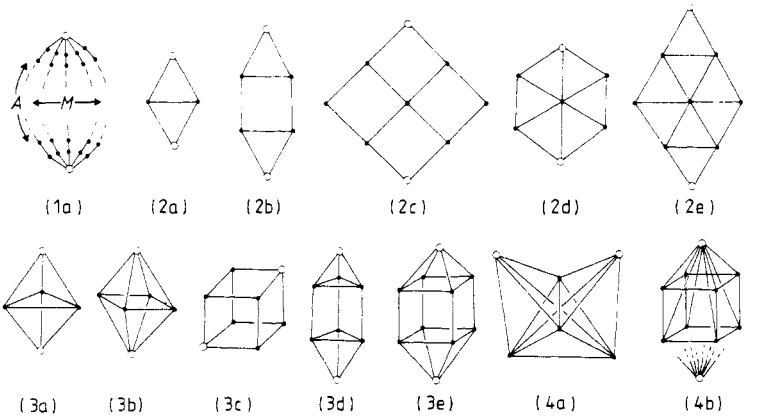

Figure 3. Some basic cells; the parametrised cell (1a) defines the Migdal-Kadanoff family of hierarchies.

natural generalisation of this for the resistance between two nodes of an nth unit: $\lambda_r \alpha (b^n)^{1-Q}$, where the connectivity $Q = \ln(q)/\ln(b)$ with q the minimum cut on the basic cell (Melrose 1983b). The role of nodes as the external surfaces of units (Griffiths and Kaufman 1982) is exposed by this. For uniform hierarchies, (23) gives $F = 2D/(D+1-Q)$ and on the Migdal-Kadanoff hierarchies (1a) with $D = 1+Q$, $F = D$ as reported in Melrose (1983a); on uniform hierarchies solving the resistance scaling problem reduces to that of (1a) with $M = q$ and $A = b$ after one has removed all bonds between equipotentials.

Observe from the table that $F > Q+1$ for low D whilst $F < Q+1$ at high D. Melrose (1983c) discusses duality on planar hierarchies; one finds that self-dual hierarchies have $F = 2$. On some families of hierarchies F tends towards finite limits as $D \to \infty$ with large members: on the hyperpyramids cells (2a, 3a, 4a, etc) $F \to 4$ as $D \to \infty$, whilst on the hypercubes (cells 2a, 3b, 4b, etc) $F \to 2$ as $D \to \infty$. From (23) it is seen that on these families λ_Z diverges with D and $\nu \to 0$ quite unlike walks on regular lattices; simply with the growing coordination of the internal sites of these cells it takes a diverging number of steps to cross the cells.

The author has not observed any stronger correlation between Ising model exponents and F over those already reported for D and Q in Melrose (1983b).

6. Random-walk renormalisation revisited

As described in § 2 on bond hierarchies with an infinite range of coordinations at and beyond a distance b^m the structure surrounding any particular mth-order site depends on the location of this site with respect to $(p > m)$-order sites; that is there is a 'structural disorder' built into such hierarchies on a scale depending on the order of the origin. The renormalisation described in § 3 can at most give the generating function for walks that return to the origin without going beyond b^m (the renormalisation naturally terminates with the decimation of an mth-order origin at the $(m-1)$th iteration). Such limitations of renormalisations calculations of functions on hierarchies have been noted previously for the single-site Greens' function (Langlois *et al* 1983).

Rammal and Toulouse (1983) proposed that on fractal lattices random-walk statistics were related to F for $F<2$. However, a basic step in their arguments does not extend to the hierarchies with an infinite range of coordinations. As pointed out in Melrose (1983b) the definition of intrinsic dimension therein does not characterise the growth of the lattice around any arbitrary site (D just characterises how the number of bonds on an nth unit depends on b^m). The number of sites available to a random walk of N steps, $A(N)$, does not obey $A(N) \propto N^{D\nu}$ (from (21), $2D\nu = F$) and hence F as found in § 4 need not be related to statistics for returns to the origin and distinct sites visited. Indeed, the Migdal-Kadanoff hierarchies (1a) provide a counter example: generating functions found by the renormalisation are those of the 1d lattice. Inspection of figure 1 reveals that the statistics for returns to an mth-order origin within b^m are simply those of returns within b^m on a one-dimensional lattice.

In a future publication the author will report Monte Carlo results for random-walk statistics on hierarchies.

Recently Melrose (1985) has reported results for the self-avoiding walk eigenvalue λ_s analogous to λ_Z here. On high-dimensional hierarchies no crossover of SAW statistics to random-walk statistics is observed; for the Migdal-Kadanoff hierarchies both λ_s and λ_Z stick to the one-dimensional lattice values for all D. The 'linear' nature of the MKH is clearly evident in figure 1.

7. Conclusions

Random-walk recursion relations for bond hierarchies were shown to be equivalent to those of harmonic models with masses scaled with the coordinations. A general result was proved for the spectral dimension, F, of harmonic models on bond hierarchies. Clear doubts as to the relevance of F to random-walk statistics on hierarchies were expressed.

Appendix. Proof that $\lambda_a = g\lambda_r$

The elements of matrix B given by (19) can be rearranged and expressed as matrix determinants in a generalisation of the Kirchoffs' solution for the resistance between two sites of an arbitrary network (Wu 1982, Stephen 1976 and references therein). One finds

$$f_1(a) = |M_{2,2}|/|M_{12,12}| \qquad f_2(a) = -|M_{2,1}|/|M_{12,12}| \tag{A1}$$

where f_1 and f_2 are defined by (18) and the notation $M_{i_1 \ldots i_p, j_1 \ldots j_q}$ is explained in the text. From (20)

$$a' = -|M_{2,2}|/|M_{2,1}|. \tag{A2}$$

When $a = 1$ all the cofactors of M as defined in (19) and (20) are equal; hence $|M_{2,2}| = -|M_{2,1}|$ and $a = 1$ is a fixed point of (A2). Now set $a = 1 + \delta$ and expand the determinants in (A2) in the diagonal perturbations $c_i\delta$. Let $\bar{M}$ stand for the matrix M in the case $a = 1$, the pure resistance problem with unit resistors. One finds

$$|M_{2,2}| = |\bar{M}_{2,2}| + \delta \sum_{j \neq 2}{}_j \, c_j |\bar{M}_{2j,2j}| + O(\delta^2),$$

$$|M_{2,1}| = |\bar{M}_{2,1}| + \delta \sum_{j \neq 1,2}{}_j \, c_j |\bar{M}_{1j,2j}| + O(\delta^2)$$

and from (A2)

$$a' = 1 + c_1|\bar{M}_{12,12}|/|\bar{M}_{2,2}| + \sum_{j\neq 1,2} c_j(|\bar{M}_{1j,2j}| + |\bar{M}_{2j,2j}|)/|\bar{M}_{2,2}| + O(\delta^2). \quad \text{(A3)}$$

Now $\bar{M}$ has the property that $\Sigma_i \bar{M}_{ij} = \Sigma_j \bar{M}_{ij} = 0\ \forall i, j$. By repeated use of this property in substitution and the identity that $|D| = |A| + |B|$ with A and B differing from D in just one row (or column), say i, such that $D_{ij} = A_{ij} + B_{ij}$, it is found that

$$\tfrac{1}{2}|\bar{M}_{12,12}| = |\bar{M}_{1j,1j}| + |\bar{M}_{1j,2j}| \qquad \forall j. \quad \text{(A4)}$$

Substitution of (A4) in (A3) and noting that with $c_1 = c_2$, $c_1 + \frac{1}{2}\Sigma_{j=3} c_j = g$, the number of bonds on the cell, one has

$$a' = 1 + \delta g \lambda_r + O(\delta^2) \quad \text{(A5)}$$

where $\lambda_r = |\bar{M}_{12,12}|/|\bar{M}_{2,2}|$ is the resistance scaling eigenvalue $R' = \lambda_r R$ and is Kirchoff's solution.

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